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The Stability of the Low-Dimensional Mixtures of Dilute Quantum Gases

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We study the stability of the dilute Bose–Fermi and Bose–Bose mixtures with repulsive interactions in one and two dimensions in terms of the renormalization group. For the Bose–Fermi mixture, we show that the uniform mixture is stable against de-mixing in the dilute limit. For the Bose–Bose mixture, we give the stability conditions in the dilute limit. As a byproduct, we also calculate the critical temperature for the superfluid phase of the two-dimensional Bose–Fermi mixture in the extremely dilute limit.

KEYWORDS: Bose–Fermi mixture, Bose–Bose mixture, renormalization group, low dimensions

1. Introduction

Ever since the achievement of Bose–Einstein condensation in the dilute gases of ultracold atoms, the study of degenerate quantum gases has become a subject attracting a lot of experimental and theoretical vigor.[1)] Due to the experimental facilities which allow control of various parameters characterizing the system, it provides an ideal laboratory for the studies of quantum many-body physics. Among them, one line of researches is to study the mixture of quantum gases. These include the mixture of bosonic and fermionic atoms (the Bose–Fermi mixture),[2)] and the mixture of two different hyperfine states of the same bosonic atoms[3)] or two different kinds of bosonic atoms[4)] (the Bose–Bose mixture). The study of the degenerate mixtures of quantum gases at lower dimensions is particularly interest and a lot of theoretical investigations are devoted to it.[5–14)] This is because the competition between the inter-species and intra-species interactions and the strong quantum fluctuations in low dimensions may introduce many interesting phenomena such as phase transition, new quantum states, and quantum phase transitions.

The present work is to study the stability of the low-dimensional Bose–Fermi (BF) and Bose–Bose (BB) mixtures with repulsive interactions in the dilute limit. For binary mixtures with mutual repulsions, the system may experience a de-mixing transition by varying the strength of the inter-species interactions. The de-mixing transition or the stability of the one-dimensional (1D) uniform mixture has been analyzed by using either the mean-field theory[5)] or the bosonization.[6,7)] The mean-field theory has the well-known problem that it can not treat properly the strong fluctuations in low dimensions. Though there is nothing wrong with the bosonization approach in one dimension, a naive extrapolation of the results obtained by the bosonization to the strong-coupling regime may sometimes lead to incorrect conclusions. For example, the bosonization approach predicts that the BF mixture will be phase separated in the strong repulsion regime.[6)] In the solvable limit, however, the Bethe-ansatz result indicates that the uniform BF mixture is stable against de-mixing irrespective of the strength of repulsions.[8)] Therefore, it is desirable to study this issue from a different approach. For the 1D single-component dilute Bose gas, it has been shown that the low-energy physics in the strong coupling regime can be well-captured by the renormalization-group (RG) method starting from a zero-density quantum critical point (QCP).[15)] Here we extend this idea to the low-dimensional mixtures with repulsions. Our results should provide a point of view complementary to the bosonization approach in the strong-coupling limit.

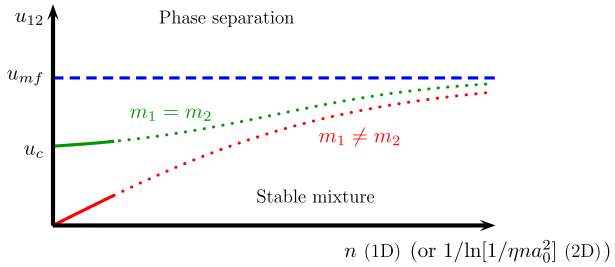

Fig. 1. (Color online) The schematic phase diagram of the 1D (2D) BB mixture with repulsions. We have taken $n_1 = n/2 = n_2$. The dashed line is the mean-field value. In 1D, the critical strength $\mathcal{U}_c^{(1)}$ at low density for $m_1 \neq m_2$ is a linear function of n. In 2D, the critical strength $\mathcal{U}_c^{(2)}$ at extremely low density for $m_1 \neq m_2$ is a linear function of $1/\ln[1/(\eta n a_0^2)]$, where $\eta = (m_2/m_1)/[2\pi(1 + m_2/m_1)^2]$. For $m_1 = m_2$, both $\mathcal{U}_c^{(1)}$ and $\mathcal{U}_c^{(2)}$ take a constant value u_c as $n \to 0$. Moreover, $\mathcal{U}_c^{(1)}$ and $\mathcal{U}_c^{(2)}$ will approach the mean-field value $u_{mf} = \sqrt{u_{11}u_{22}}$ at large density. In between, we assume that they are smooth functions. In the SU(2) symmetrical limit, $u_c = u_{mf}$.

In one dimension, there is an interacting zero-density fixed point which controls the low-energy physics of the 1D uniform mixture in the strongly repulsive limit. Similar to the single-component dilute Bose gas, this nontrivial fixed point can be captured by the ϵ expansion, where $\epsilon = 2 - d$. For the two-dimensional (2D) mixtures, all couplings are marginally irrelevant in the sense of RG. Thus, we expect that the stability conditions given by the mean-field theory are correct to the leading order. Our main results are as follows: (i) We find that the 1D uniform BF mixture in the dilute limit, i.e., the strong-coupling regime, is always stable against de-mixing for any finite strength of repulsive interactions. This is consistent with Bethe-ansatz result in the solvable limit.[8)] And our analysis extends this result to the region beyond the solvable limit. (ii) We derive the stability condition of the 1D uniform BB mixture in the strong-coupling regime (Fig. 1), which is different from the bosonization result. (iii) In the extremely dilute limit, we find that the interactions for the 2D mixtures are strongly renormalized so that the stability conditions obtained by the

mean-field theory are changed considerably. (iv) We also calculate the critical temperature for the 2D BF mixture in the extremely dilute limit [eq. (26)] to investigate one aspect of the effects arising from the inclusion of fermions. Although the stability of the mixtures with attractions is also an interesting issue,[10] it is beyond the scope of the present work. This is because the fixed-point structure for attractions is different from the one for repulsions, i.e., the former exhibits a runaway RG flow, and thus our approach cannot give definite results. Finally, we notice that a similar RG analysis of the stability of the dilute BB mixture in low dimensions was also carried out in a recent paper.[16] The results are different from ours in certain aspects. We will address the origin of this discrepancy at the end of the paper.

The rest of the paper is devoted to the detailed analysis that leads to the above conclusions. Sections 2 and 3 are about the BF and BB mixtures, respectively. We compare our results with the previous work in the last section. An appendix about the mean-field treatment of the BF mixture in d dimensions is provided for reference.

2. The Bose–Fermi Mixture

To study the mixture of bosons and spinless fermions in the dilute limit, we start with the action

$$\begin{aligned} S = &\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\, \psi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m_\mathrm{f}} - \mu_\mathrm{f}\right)\psi \\ &+ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x\, \phi^\dagger \left(\partial_\tau - \frac{\nabla^2}{2m_\mathrm{b}} - \mu_\mathrm{b}\right)\phi \\ &+ \int_0^\beta \mathrm{d}\tau \int \mathrm{d}^d x \left(\frac{g_\mathrm{bb}}{2}|\phi|^4 + g_\mathrm{bf}|\phi|^2|\psi|^2\right), \end{aligned} \tag{1}$$

where $\beta = 1/T$. (We will take $\hbar = 1$ and $k_\mathrm{B} = 1$.) Here ϕ and ψ are the fields describing the bosonic and fermionic atoms, respectively. m_f, μ_f and m_b, μ_b are the masses and chemical potentials for the bosons and fermions, respectively. The couplings g_bf and g_bb are related to the scattering lengths a_bf and a_bb through the relations $g_\mathrm{bf} = 2\omega_\perp a_\mathrm{bf}$ and $g_\mathrm{bb} = 2\omega_\perp a_\mathrm{bb}$ for the 1D trap,[17] and $g_\mathrm{bf} = \sqrt{4\pi(m_\mathrm{b}+m_\mathrm{f})\omega_\perp/m_\mathrm{b}m_\mathrm{f}}\,a_\mathrm{bf}$ and $g_\mathrm{bb} = \sqrt{8\pi\omega_\perp/m_\mathrm{b}}\,a_\mathrm{bb}$ for a 2D trap,[18] where $\omega_\perp$ is the transverse confining frequency. We shall take $g_\mathrm{bb}, g_\mathrm{bf} > 0$ to avoid the possible formation of bound states. We also assume that

$$\mu_\mathrm{b}, \mu_\mathrm{f} \ll \frac{1}{m_\alpha a_0^2}, \tag{2}$$

where $\alpha = \mathrm{b}, \mathrm{f}$, and a_0 is the average range of interactions between atoms, which can be regarded as the short-distance cutoff of the action S. We will see later that this condition amounts to the dilute limit.

The action S [eq. (1)] has a QCP at $\mu = 0$, $T = 0$, and $g_\mathrm{bb} = 0 = g_\mathrm{bf}$ (the Gaussian fixed point). In the dilute limit, this Gaussian fixed point may be an appropriate departure point to study the low-energy physics of this system. To proceed, we shall employ the RG method. The strategy is as the following. We first analyze the fixed-point structure of the theory. Then we integrate the RG equations to the scale where the correlation length becomes of $O(1)$ in the unit of the cutoff of the resulting renormalized Hamiltonain. At this scale the quantum fluctuations have been taken into account (within the ϵ expansion), and thus we may apply the mean-field theory to the renormalized Hamiltonian to study the stability of the mean-field solution corresponding to the uniform mixture. In terms of the solutions of RG equations, we may express the stability conditions in terms of the physical quantities.

2.1 Renormalization-group theory

To proceed, we first make a change of variables:

$$\boldsymbol{x} = a_0\tilde{\boldsymbol{x}}, \quad \tau = ma_0^2\tilde{\tau}, \quad \psi = a_0^{-d/2}\tilde{\psi}, \quad \phi = a_0^{-d/2}\tilde{\phi},$$

such that $\tilde{\boldsymbol{x}}$, $\tilde{\tau}$, $\tilde{\psi}$, $\tilde{\phi}$ all become dimensionless, where m is a quantity carrying the dimension of mass. Then, the action S [eq. (1)] at $T = 0$ can be written as

$$\begin{aligned} S = &\int \mathrm{d}\tilde{\tau} \int \mathrm{d}^d\tilde{x}\, \tilde{\psi}^\dagger \left(\partial_{\tilde{\tau}} - \frac{\nabla^2}{2\tilde{m}_\mathrm{f}} - r_\mathrm{f}\right)\tilde{\psi} \\ &+ \int \mathrm{d}\tilde{\tau} \int \mathrm{d}^d\tilde{x}\, \tilde{\phi}^\dagger \left(\partial_{\tilde{\tau}} - \frac{\nabla^2}{2\tilde{m}_\mathrm{b}} - r_\mathrm{b}\right)\tilde{\phi} \\ &+ \int \mathrm{d}\tilde{\tau} \int \mathrm{d}^d\tilde{x} \left(\frac{\lambda_\mathrm{bb}}{2K_d\tilde{m}_\mathrm{b}}|\tilde{\phi}|^4 + \frac{\lambda_\mathrm{bf}}{K_d\tilde{m}_\mathrm{bf}}|\tilde{\phi}|^2|\tilde{\psi}|^2\right), \end{aligned}$$

where $m_\mathrm{bf} = 2m_\mathrm{b}m_\mathrm{f}/(m_\mathrm{b}+m_\mathrm{f})$, $\tilde{m}_\mathrm{b} = m_\mathrm{b}/m$, $\tilde{m}_\mathrm{f} = m_\mathrm{f}/m$, $\tilde{m}_\mathrm{bf} = m_\mathrm{bf}/m$, and

$$\begin{aligned} r_\mathrm{f} &= m\mu_\mathrm{f}a_0^2, & r_\mathrm{b} &= m\mu_\mathrm{b}a_0^2, \\ \lambda_\mathrm{bb} &= K_d m_\mathrm{b} g_\mathrm{bb} a_0^\epsilon, & \lambda_\mathrm{bf} &= K_d m_\mathrm{bf} g_\mathrm{bf} a_0^\epsilon, \end{aligned} \tag{3}$$

are all dimensionless parameters. In the above, we define $K_d = 2/[(4\pi)^{d/2}\Gamma(d/2)]$ ($\Gamma(x)$ is the Gamma function) and $\epsilon = 2 - d$. We will see later that m always appear in the physical quantities in the guise of ma_0^2. Therefore, for universal quantities which do not depend on a_0 explicitly, the dependence on m is completely dropped out. The determination of the short-distance energy scale $1/(ma_0^2)$ relies on the comparison with exact solutions or experimental results.

To perform the RG transformations, we decompose the fields $\tilde{\psi}$ and $\tilde{\phi}$ into $\tilde{\psi} = \tilde{\psi}_< + \tilde{\psi}_>$ and $\tilde{\phi} = \tilde{\phi}_< + \tilde{\phi}_>$, where $\tilde{\psi}_>$ ($\tilde{\phi}_>$) consists of the Fourier components of $\tilde{\psi}$ ($\tilde{\phi}$) with $\Lambda e^{-l} < |\boldsymbol{k}| < \Lambda$, and $\tilde{\psi}_<$ ($\tilde{\phi}_<$) consists of the Fourier components of $\tilde{\psi}$ ($\tilde{\phi}$) with $|\boldsymbol{k}| < \Lambda e^{-l}$. Here Λ is the (dimensionless) UV cutoff for the momentum and $l > 0$ is the scaling parameter. By integrating out $\tilde{\psi}_>$ and $\tilde{\phi}_>$ to the one-loop order and then rescaling the momentum and frequency by $\boldsymbol{k} \to e^{-l}\boldsymbol{k}$ and $\omega \to e^{-2l}\omega$, we obtain an effective action for the slow modes. By ignoring the irrelevant operators in the sense of RG, this effective action has an identical form to the original action S but with different values of the parameters $r_\alpha(l)$, $\lambda_\mathrm{bb}(l)$, and $\lambda_\mathrm{bf}(l)$, which can be regarded as the corresponding parameters at the momentum scale Λe^{-l}. By comparing the effective action of the slow modes with the original action and considering an infinitesimal value of l, we get the one-loop RG equations

$$\frac{\mathrm{d}r_\alpha}{\mathrm{d}l} = 2r_\alpha, \tag{4}$$

and

$$\begin{aligned} \frac{\mathrm{d}\lambda_\mathrm{bb}}{\mathrm{d}l} &= \epsilon\lambda_\mathrm{bb} - \frac{\lambda_\mathrm{bb}^2}{1 - 2\tilde{m}_\mathrm{b}r_\mathrm{b}}, \\ \frac{\mathrm{d}\lambda_\mathrm{bf}}{\mathrm{d}l} &= \epsilon\lambda_\mathrm{bf} - \frac{\lambda_\mathrm{bf}^2}{1 - \tilde{m}_\mathrm{bf}(r_\mathrm{b} + r_\mathrm{f})}, \end{aligned} \tag{5}$$

where the initial values $r_\alpha(0)$, $\lambda_{\rm bb}(0)$, and $\lambda_{\rm bf}(0)$ are given by eq. (3). Since we are only interested in the dilute limit so that $r_\alpha(l) \ll 1$ before scaling stops, eq. (5) can be approximated as[19]

$$\frac{{\rm d}\lambda}{{\rm d}l} = \epsilon\lambda - \lambda^2, \tag{6}$$

where λ represents either $\lambda_{\rm bb}$ or $\lambda_{\rm bf}$.

If $d \neq 2$, eqs. (4) and (6) will have two fixed points: (i) the Gaussian fixed point $(r_\alpha, \lambda_{\rm bb}, \lambda_{\rm bf}) = (0,0,0)$ and (ii) the interacting zero-density fixed point $(r_\alpha, \lambda_{\rm bb}, \lambda_{\rm bf}) = (0, \lambda_*, \lambda_*)$ with $\lambda_* = \epsilon$. On the other hand, for $d = 2$, the two fixed points merge into one — the Gaussian fixed point. Since $g_{\rm bb}, g_{\rm bf} > 0$, in the dilute limit, the zero-density fixed point will control the low energy physics in one dimension, whereas in two dimensions it is the Gaussian fixed point which will govern the low energy physics. Since the low density implies strong couplings in one dimension, this nontrivial zero-density fixed point will describe the strong-coupling physics.

The solution for eq. (6) in $d < 2$ is of the form

$$\lambda_{\rm bb}(l) = \frac{\lambda_*}{1 + c_{\rm bb} e^{-\epsilon l}}, \quad \lambda_{\rm bf}(l) = \frac{\lambda_*}{1 + c_{\rm bf} e^{-\epsilon l}}, \tag{7}$$

with

$$c_{\rm bb} = \frac{\lambda_*}{K_d m_{\rm b} g_{\rm bb} a_0^\epsilon} - 1, \quad c_{\rm bf} = \frac{\lambda_*}{K_d m_{\rm bf} g_{\rm bf} a_0^\epsilon} - 1,$$

while in $d = 2$ it takes the form

$$\lambda_{\rm bb}(l) = \frac{K_2 m_{\rm b} g_{\rm bb}}{1 + K_2 m_{\rm b} g_{\rm bb} l}, \quad \lambda_{\rm bf}(l) = \frac{K_2 m_{\rm bf} g_{\rm bf}}{1 + K_2 m_{\rm bf} g_{\rm bf} l}. \tag{8}$$

Scaling stops at $l = l_*$ where eq. (6) can not be applied any more. This occurs when $r_{\rm b}(l_*) + r_{\rm f}(l_*) = 1$,[20] yielding

$$e^{l_*} = \frac{1}{\sqrt{m(\mu_{\rm b} + \mu_{\rm f}) a_0^2}}. \tag{9}$$

From eq. (9), we see that the condition (2) implies that $e^{l_*} \gg 1$.

The uniform mixture corresponds to the state with non-vanishing values of $\rho_{\rm b} = \langle|\tilde{\phi}(l_*)|^2\rangle$ and $\rho_{\rm f} = \langle|\tilde{\psi}(l_*)|^2\rangle$. Within the framework of the RG, the phase boundary of the uniform mixture is determined as follows: First of all, we calculate the effective potential of the order parameter $\rho_{\rm b}$ at given μ_α:

$$f = \frac{e^{-(d+2)l_*}}{m a_0^{d+2}} \tilde{f}(l_*), \tag{10}$$

where $\tilde{f}(l_*)$ is the effective potential of the renormalized Hamiltonian at $l = l_*$. Next, as a function of $\rho_{\rm b}$, the equilibrium state is determined by the absolute minimum of f at fixed μ_α. Since the prefactor in eq. (10) is always positive and does not depend on $\rho_{\rm b}$, the absolute minimum of f corresponds to the absolute minimum of $\tilde{f}(l_*)$. Therefore, to search for the absolute minimum of f, it suffices to consider $\tilde{f}(l_*)$. In particular, the stability of the uniform mixture can be determined from $\tilde{f}(l_*)$. Because the coupling constants at $l = l_*$ are small (in the sense of the ϵ expansion for $d < 2$), we may calculate $\tilde{f}(l_*)$ in terms of the perturbation theory in $\lambda(l_*)$. The leading order result will given by the mean-field theory. In the following, we will employ this RG-improved mean-field theory to study the stability of the uniform mixture. To avoid the confusion, we emphasize again that f is a functional of the order parameter $\rho_{\rm b}$ at given μ_α, and its absolute minimum determines the thermodynamical relation between $\rho_{\rm b}$ and μ_α and the associated stability conditions. The free energy density Ω is obtained by inserting the resulting value of $\rho_{\rm b}$ into f, i.e., $\Omega = f(\mu_\alpha, \rho_{\rm b}(\mu_\alpha))$.

2.2 *The uniform mixture in one dimension*

We first study the stability of the 1D uniform mixture. From eqs. (A·2) and (A·3), to the leading order, $\rho_{\rm b}$ and $\rho_{\rm f}$ are determined by the mean-field equations

$$\frac{\lambda_{\rm bf}^*}{K_d \tilde{m}_{\rm bf}} \rho_{\rm f} = r_{\rm b}^* - \frac{\lambda_{\rm bb}^*}{K_d \tilde{m}_{\rm b}} \rho_{\rm b},$$

and

$$\rho_{\rm f} = \frac{K_d}{d} \left[2\tilde{m}_{\rm f} r_{\rm f}^* - \frac{(m_{\rm b} + m_{\rm f})\lambda_{\rm bf}^*}{K_d m_{\rm b}} \rho_{\rm b} \right]^{d/2},$$

where $r_\alpha^* = r_\alpha(l_*)$, $\lambda_{\rm bb}^* = \lambda_{\rm bb}(l_*)$, and $\lambda_{\rm bf}^* = \lambda_{\rm bf}(l_*)$. In the dilute limit, we may simply set $\lambda_{\rm bb}^* = \lambda_* = \lambda_{\rm bf}^*$. Hence, by solving the two equations within the ϵ expansion, we get

$$\begin{aligned} \rho_{\rm b} &= \frac{K_d \tilde{m}_{\rm b}}{\lambda_*} r_{\rm b}^* - \frac{K_d}{d} (\tilde{m}_{\rm b} + \tilde{m}_{\rm f}) r_{\rm f}^* + O(\epsilon), \\ \rho_{\rm f} &= \frac{2K_d \tilde{m}_{\rm f}}{d} r_{\rm f}^* - \frac{K_d}{d} (\tilde{m}_{\rm b} + \tilde{m}_{\rm f}) r_{\rm b}^* + O(\epsilon). \end{aligned} \tag{11}$$

We still have to relate μ_α (or $r_{\rm b}^*$ and $r_{\rm f}^*$) to the boson density $n_{\rm b}$ and the fermion density $n_{\rm f}$. This can be achieved through the scaling equations

$$\begin{aligned} n_{\rm b} &= e^{-dl_*} n_{\rm b}(l_*) = e^{-dl_*} a_0^{-d} \rho_{\rm b}, \\ n_{\rm f} &= e^{-dl_*} n_{\rm f}(l_*) = e^{-dl_*} a_0^{-d} \rho_{\rm f}. \end{aligned} \tag{12}$$

Inserting eqs. (9) and (11) into eq. (12) and noticing that $r_\alpha^* = \mu_\alpha/(\mu_{\rm b} + \mu_{\rm f})$, we find that

$$e^{l_*} = \left[\frac{\lambda_*}{2K_d \tilde{m}_{\rm f}} (1 + 3\xi) n_{\rm b} a_0^d + \frac{d(d+2)}{4K_d \tilde{m}_{\rm f}} n_{\rm f} a_0^d \right]^{-1/d}, \tag{13}$$

and

$$\begin{aligned} r_{\rm b}^* &= \frac{2\xi n_{\rm b} + (1+\xi) n_{\rm f}}{(1+3\xi) n_{\rm b} + [d(d+2)/(2\lambda_*)] n_{\rm f}}, \\ r_{\rm f}^* &= \frac{(1+\xi) n_{\rm b} + [d(d+2)/(4\lambda_*)] n_{\rm f}}{(1+3\xi) n_{\rm b} + [d(d+2)/(2\lambda_*)] n_{\rm f}}, \end{aligned} \tag{14}$$

where $\xi = m_{\rm f}/m_{\rm b}$. By comparing eq. (9) with eq. (13), we see that the condition (2) indeed amounts to the diluteness condition $n_\alpha a_0^d \ll 1$.

Now we are able to study the stability condition for the 1D uniform mixture. From eq. (A·5), the stability condition can be written as

$$\rho_{\rm f}^{\epsilon/d} > \left(\frac{K_d}{d} \right)^{\epsilon/d} \frac{m_{\rm b} m_{\rm f} (\lambda_{\rm bf}^*)^2}{m_{\rm bf}^2 \lambda_{\rm bb}^*}. \tag{15}$$

Inserting eqs. (7), (12), and (13) into eq. (15), the stability condition can be written as

$$1 > \lambda_* G\left(\frac{n_{\rm b}}{n_{\rm f}} \right) \left[\frac{1 + c_{\rm bb} e^{-\epsilon l_*}}{(1 + c_{\rm bf} e^{-\epsilon l_*})^2} \right], \tag{16}$$

where

$$G(x) = \frac{(1+\xi)^2}{4\xi} \left[\frac{d+2}{2} + \frac{\lambda_*}{d} (1+3\xi) x \right]^{\epsilon/d}.$$

We may use eq. (16) to determine the critical strength $g_{\rm bf}^{(c)}(\epsilon)$ of $g_{\rm bf}$ at given $g_{\rm bb}$, $n_{\rm b}$, and $n_{\rm f}$ by assuming that $\epsilon \ll 1$. Then, the critical strength of $g_{\rm bf}$ in $d = 1$ can be estimated by setting $\epsilon = 1$. To proceed, we notice that $e^{-\epsilon l_*}$ must be treated as an exponentially small quantity which can not be expanded as a power series in ϵ. This is because eq. (16) are obtained by the double expansion in $e^{-\epsilon l_*}$ and ϵ.[21] Moreover, $G(n_{\rm b}/n_{\rm f}), c_{\rm bb} = O(1) \ll 1/\epsilon$ for given $g_{\rm bb}$, $n_{\rm b}$, and $n_{\rm f}$. Keeping these in mind, we find that

$$K_d m_{\rm bf} g_{\rm bf} a_0^{\epsilon} > -\epsilon e^{-\epsilon l_*}\left[1 + \sqrt{\epsilon G\left(\frac{n_{\rm b}}{n_{\rm f}}\right)}\right]. \tag{17}$$

Since the value of the R.H.S. in inequality (17) is negative, it is always satisfied as long as $g_{\rm bf} > 0$. This means that the uniform mixture in $2 - \epsilon$ dimensions is always stable for any finite strength of $g_{\rm bf} > 0$. By extrapolating this result to $\epsilon = 1$, our analysis suggests that the 1D uniform mixture may be stable against de-mixing for any finite strength of repulsions.

The action S [eq. (1)] can be exactly solved by the Bethe ansatz under the conditions $m_{\rm b} = m_{\rm f}$ and $g_{\rm bb} = g_{\rm bf}$.[8] In the solvable limit, the uniform BF mixture is shown to be stable against de-mixing. This result is correctly reproduced in our approach, and is extended to the region beyond the solvable limit.

2.3 *The uniform mixture in two dimensions*

Now we turn to the 2D case. Here we will study two problems: (i) the stability condition for the uniform mixture and (ii) the effect of fermions on the critical temperature $T_{\rm c}$ of the superfluid (SF) phase.

We first examine the stability condition. Following from eq. (15), the stability condition in $d = 2$ becomes

$$(\lambda_{\rm bf}^*)^2 < \frac{2m_{\rm bf}}{m_{\rm b} + m_{\rm f}}\lambda_{\rm bb}^*. \tag{18}$$

The diluteness condition only requires that $e^{l_*} \gg 1$. On account of the logarithmic RG flow for the coupling constants in $d = 2$, we have two situations:

When $e^{l_*} \gg 1$ but $l_* = O(1)$, we may set the coupling constants by their bare values in eq. (18), i.e., $\lambda_{\rm bb}^* = K_2 m_{\rm b} g_{\rm bb}$ and $\lambda_{\rm bf}^* = K_2 m_{\rm bf} g_{\rm bf}$. Thus, the stability condition in this case is

$$g_{\rm bf}^2 < \frac{2\pi g_{\rm bb}}{m_{\rm f}}. \tag{19}$$

This is just the mean-field result.

On the other hand, when $l_* \gg 1$, which is referred to as the extremely dilute limit, we have to insert eq. (8) into eq. (18), yielding

$$l_* > \frac{(m_{\rm b} + m_{\rm f})^2}{4m_{\rm b}m_{\rm f}}\left[1 - \frac{8\pi}{(m_{\rm b} + m_{\rm f})g_{\rm bf}}\right]. \tag{20}$$

We still have to determine l_* as a function of n_α. For $l_* \gg 1$, we may simply set $\lambda_{\rm bb}^* = 1/l_* = \lambda_{\rm bf}^*$. The rest of the procedure is similar to the 1D case, and the result is

$$l_* \approx \frac{1}{2}\ln\left(\frac{1 + 3\xi}{2\pi n_{\rm f} a_0^d}\right),$$

to the leading order in the $1/l_*$ expansion. Consequently, the extremely dilute limit is given by the condition

$$\ln\left(\frac{1}{2\pi n_{\rm f} a_0^d}\right) \gg 1. \tag{21}$$

Since the right hand side of eq. (20) is of $O(1)$, it is automatically satisfied as long as eq. (21) is obeyed. Therefore, we conclude that in the extremely dilute limit the uniform mixture is stable against demixing for any finite strength of repulsions. In other words, the quantum fluctuations stabilize the uniform mixture with repulsive interactions in the extremely dilute limit.

In the parameter space where the uniform mixture is stable, the system will becomes a SF phase through the KT-type transition as lowering the temperature T. Here we would like to calculate the critical temperature $T_{\rm c}$ for the SF phase in terms of the RG. Now there are three relevant parameters $r_{\rm b}(l)$, $r_{\rm f}(l)$, and t_l in the sense of RG, where the dimensionless temperature t_l is defined as $t_l = t_0 e^{2l}$ with $t_0 = mTa_0^2$. To proceed, we will assume that $|\mu_\alpha| \ll T_{\rm c} \ll 1/(m_\alpha a_0^2)$. In this parameter regime, it is t_l which reaches $O(1)$ first under the RG transformations. Hence, we run the RG equations to the scale $l = \tilde{l}$ such that $t_{\tilde{l}} = 1$. Accordingly, $\tilde{l}$ is given by

$$\tilde{l} = \frac{1}{2}\ln\left(\frac{1}{mTa_0^2}\right). \tag{22}$$

Before l reaches $\tilde{l}$, $t_l \ll 1$, and thus we may approximate the RG equations for $r_\alpha(l)$, $\lambda_{\rm bb}(l)$, and $\lambda_{\rm bf}(l)$ as the ones at $T = 0$. This may introduce multiplicative errors of order unity coming from the imprecise treatment of the regime $t_l \sim 1$, but makes the analysis simpler. At $l = \tilde{l}$, the action S becomes

$$\begin{aligned} S = &\int_0^1 {\rm d}\tilde{\tau}\int {\rm d}^2\tilde{x}\,\tilde{\psi}^\dagger\left(\partial_{\tilde{\tau}} - \frac{\nabla^2}{2\tilde{m}_{\rm f}} - \tilde{r}_{\rm f}\right)\tilde{\psi} \\ &+ \int_0^1 {\rm d}\tilde{\tau}\int {\rm d}^2\tilde{x}\,\tilde{\phi}^\dagger\left(\partial_{\tilde{\tau}} - \frac{\nabla^2}{2\tilde{m}_{\rm b}} - \tilde{r}_{\rm b}\right)\tilde{\phi} \\ &+ \int_0^1 {\rm d}\tilde{\tau}\int {\rm d}^2\tilde{x}\left(\frac{\tilde{\lambda}}{2K_2\tilde{m}_{\rm b}}|\tilde{\phi}|^4 + \frac{\tilde{\lambda}}{K_2\tilde{m}_{\rm bf}}|\tilde{\phi}|^2|\tilde{\psi}|^2\right), \end{aligned}$$

where $\tilde{r}_\alpha = r_\alpha(\tilde{l}) = \mu_\alpha/T$ and $\tilde{\lambda} = 1/\tilde{l}$. Here we assume that $\tilde{l} \gg 1$ so that $\lambda_{\rm bb}(\tilde{l}) \approx \tilde{\lambda}$ and $\lambda_{\rm bf}(\tilde{l}) \approx \tilde{\lambda}$. Since $\tilde{\lambda} \ll 1$, a perturbative expansion in $\tilde{\lambda}$ is reliable.

The field $\tilde{\phi}$ may be considered the order parameter of the SF phase. Hence, the critical temperature can be determined by the equation

$$\tilde{r}_{\rm b} + \tilde{\Sigma} = 0, \tag{23}$$

where $\tilde{\Sigma}$ is the self-energy of $\tilde{\phi}$. To the one-loop order, $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma} \approx 2\tilde{\lambda}\ln\tilde{r}_{\rm b}.$$

Here, we keep only the leading term in the small $\tilde{r}_\alpha$ expansion. Inserting this result into eq. (23), we obtain

$$\tilde{r}_{\rm b} \approx -2\tilde{\lambda}\ln(2\tilde{\lambda}). \tag{24}$$

We have to express $\tilde{r}_{\rm b}$ as a function of n_α and T. The procedure is similar to the 1D case, and the result is

$$\tilde{r}_{\rm b} = \frac{\tilde{\lambda}}{K_2 m_{\rm b} T}n_{\rm b} + \frac{\tilde{\lambda}}{K_2 m_{\rm bf} T}n_{\rm f}.$$

Inserting this expression and using eq. (22), we find

$$\frac{\pi}{m_b T}\left(n_b+\frac{1+\xi}{2\xi}n_f\right)=\ln\left[\frac{1}{4}\ln\left(\frac{1}{mTa_0^2}\right)\right]. \quad (25)$$

From eq. (25), T_c is given by

$$T_c=\frac{\pi[n_b+(1+\xi)/(2\xi)n_f]}{m_b\ln\left(\frac{1}{4}\ln\left\{\frac{\tilde{m}_b}{\pi[n_b+(1+\xi)/(2\xi)n_f]a_0^2}\right\}\right)}. \quad (26)$$

By setting $n_f=0$, eq. (26) reduces to the critical temperature for a single-component dilute Bose gas.[22] Equation (26) suggests that the inclusion of fermions increases the critical temperature.

3. The Bose–Bose Mixture

The dilute BB mixture can be described by the action

$$S=\int_0^\beta d\tau\int d^dx\left[\sum_{\alpha=1,2}\Psi_\alpha^\dagger\left(\partial_\tau-\frac{\nabla^2}{2m_\alpha}-\mu_\alpha\right)\Psi_\alpha+U\right], \quad (27)$$

where

$$U=\frac{1}{2}\sum_{\alpha=1,2}u_{\alpha\alpha}|\Psi_\alpha|^4+u_{12}|\Psi_1|^2|\Psi_2|^2.$$

Here Ψ_α are the fields describing the bosonic atoms with masses m_α. The couplings $u_{\alpha\beta}$ are related to the scattering lengths $a_{\alpha\beta}$ through the relations $u_{12}=2\omega_\perp a_{12}$ and $u_{\alpha\alpha}=2\omega_\perp a_{\alpha\alpha}$ for the 1D trap, and $u_{12}=\sqrt{4\pi(m_1+m_2)\omega_\perp/m_1m_2}a_{12}$ and $u_{\alpha\alpha}=\sqrt{8\pi\omega_\perp/m_\alpha}a_{\alpha\alpha}$ for a 2D trap, where $\omega_\perp$ is the transverse confining frequency. We shall take $u_{\alpha\beta}>0$.

By integrating out the fast modes, the one-loop RG equations in the dilute limit are completely identical to eqs. (4) and (6), with the initial conditions

$$r_\alpha(0)=(m_1+m_2)\mu_\alpha a_0^2,\quad \lambda_{\alpha\beta}(0)=K_d m_{\alpha\beta}u_{\alpha\beta}a_0^\epsilon.$$

Here $m_{\alpha\beta}=2m_\alpha m_\beta/(m_1+m_2)$. Hence, the fixed-point structure is identical to the one for the BF mixture. The solution of the RG equation for $\lambda_{\alpha\beta}(l)$ in $d=1$ is of the form

$$\lambda_{\alpha\beta}(l)=\frac{\lambda_*}{1+c_{\alpha\beta}e^{-l}}, \quad (28)$$

where

$$c_{\alpha\beta}=\frac{\lambda_*}{K_1 m_{\alpha\beta}a_0u_{\alpha\beta}}-1,$$

while in $d=2$ it is given by

$$\lambda_{\alpha\alpha}(l)=\frac{K_2 m_{\alpha\beta}u_{\alpha\beta}}{1+K_2 m_{\alpha\beta}u_{\alpha\beta}l}. \quad (29)$$

Scaling stops when $l=l_*$, where $r_1(l_*)+r_2(l_*)=1$, yielding

$$e^{l_*}=\frac{1}{\sqrt{(m_1+m_2)(\mu_1+\mu_2)a_0^2}}. \quad (30)$$

By a procedure similar to the Bose–Fermi mixture, l_* in $d=1$ can be expressed as

$$e^{l_*}=\frac{m_{12}/(\pi\lambda_*)}{a_0[(m_1+3m_2)n_1+(3m_1+m_2)n_2]}, \quad (31)$$

in the dilute limit. On the other hand, in $d=2$, l_* takes the form

$$l_*=\ln\sqrt{\frac{m_{12}/(2\pi)}{[(m_1+3m_2)n_1+(3m_1+m_2)n_2]a_0^2}}, \quad (32)$$

in the extremely dilute limit. Here the extremely dilute limit is defined by $l_*\gg 1$. (The diluteness condition only guarantees that $e^{l_*}\gg 1$.)

We first use the above results to study the 1D uniform mixture. Following the argument similar to the dilute BF mixture, the stability condition for the uniform mixture can be obtained from a mean-field analysis on the renormalized Hamiltonian, yielding

$$\frac{4m_1m_2}{(m_1+m_2)^2}\lambda_{11}^*\lambda_{22}^*>(\lambda_{12}^*)^2, \quad (33)$$

where $\lambda_{\alpha\beta}^*=\lambda_{\alpha\beta}(l_*)$. It is clear that the fixed-point Hamiltonian (with $\lambda_{\alpha\beta}=\lambda_*$) cannot satisfy eq. (33). This means that the fixed-point Hamiltonian does not describe a uniform mixture, and thus we cannot take the continuum limit directly, as we have done for the BF mixture.

To proceed, we insert eq. (28) into eq. (33) and use eq. (31). Then, in the dilute limit ($e^{l_*}\gg 1$), the stability condition can be expressed as

$$0<u_{12}<\mathcal{U}_c^{(1)}, \quad (34)$$

where

$$\mathcal{U}_c^{(1)}=\frac{u_c}{1+\frac{m_1m_2(m_1-m_2)^2u_c}{2\pi^2(m_1+m_2)^2[(m_1+3m_2)n_1+(3m_1+m_2)n_2]}},$$

is the critical strength of u_{12} in $d=1$, and

$$u_c=\frac{2m_{12}u_{11}u_{22}}{m_1u_{11}+m_2u_{22}}.$$

In the above, we have set $\lambda_*=\epsilon=1$. From the expression for $\mathcal{U}_c^{(1)}$, we see that for the mixtures consisting of two different kinds of bosons, i.e., $m_1\neq m_2$, the critical strength $\mathcal{U}_c^{(1)}$ is a linear function of boson densities

$$\mathcal{U}_c^{(1)}\approx\frac{2\pi^2(m_1+m_2)^2}{m_1m_2(m_1-m_2)^2}[(m_1+3m_2)n_1+(3m_1+m_2)n_2],$$

in the dilute limit, which may be much smaller than u_c. Moreover, $\mathcal{U}_c^{(1)}$ is is insensitive to the values of $u_{\alpha\alpha}$. On the other hand, for the mixtures composed of two different hyperfine states of the same boson, i.e., $m_1=m_2$, $\mathcal{U}_c^{(1)}$ reduces to

$$\mathcal{U}_c^{(1)}=\frac{2u_{11}u_{22}}{u_{11}+u_{22}},$$

in the dilute limit, which is independent of the boson densities.

The stability conditions we obtained are valid only in the strong-coupling regime. We notice that they are different from the result obtained from bosonization, which gave $\mathcal{U}_c^{(1)}=\pi^2\sqrt{n_1n_1/(m_1m_2)}$. The study of the 1D BB mixture with the exchange symmetry, i.e., $m_1=m_2$ and $u_{11}=u_{22}=U$, by the finite-size density-matrix RG (FSDMRG) indicates that the phase separation occurs for all boson densities when $u_{12}>U$.[14] Our conclusion is consistent with this result when the system possesses the exchange symmetry.

Next, we turn into the 2D case. Due to the logarithmic RG flow of the coupling constants, we have two situations:

(i) When $e^{l_*} \gg 1$ but $l_* = O(1)$, we may take $\lambda^*_{\alpha\beta}$ by their initial values, i.e., $\lambda^*_{\alpha\beta} \approx K_2 m_{\alpha\beta} u_{\alpha\beta}$. Inserting these into eq. (33) gives rise to the mean-field result

$$0 < u_{12} < \sqrt{u_{11}u_{22}}. \tag{35}$$

(ii) In the extremely dilute limit, i.e., $l_* \gg 1$, we insert eq. (29) into eq. (33). Then, the stability condition in this situation becomes

$$0 < u_{12} < \mathcal{U}^{(2)}_c, \tag{36}$$

where

$$\mathcal{U}^{(2)}_c = \frac{u_c}{1 + \dfrac{(m_1 - m_2)^2 u_c}{16\pi(m_1 + m_2)} l_*},$$

and l_* is given by eq. (32). We see that for the mixtures composed of two different hyperfine states of the same boson, i.e., $m_1 = m_2$, the critical strength is given by $\mathcal{U}^{(2)}_c = u_c$, which is independent of the boson densities, similar to the 1D case. On the other hands, for the mixtures consisting of two different kinds of bosons, i.e., $m_1 \neq m_2$, $\mathcal{U}^{(2)}_c \approx 16\pi(m_1 + m_2)/[(m_1 - m_2)^2 l_*]$. That is, it is a function of boson densities, which is insensitive to the values of $u_{\alpha\alpha}$. However, the dependence of $\mathcal{U}^{(2)}_c$ on the boson densities is much weaker than the 1D case due to the presence of the logarithm. This is a characteristic of the 2D Bose gas.

A schematic phase diagram for the 1D and 2D BB mixtures with repulsions is shown in Fig. 1, where we have taken $n_1 = n/2 = n_2$. For $m_1 \neq m_2$, $\mathcal{U}^{(1)}_c$ is a linear function of n at low density and $\mathcal{U}^{(2)}_c$ is a linear function of $1/\ln[1/(\eta n a_0^2)]$ at extremely low density with

$$\eta = \frac{m_2/m_1}{2\pi(1 + m_2/m_1)^2}.$$

On the other hand, $\mathcal{U}^{(1)}_c = u_c = \mathcal{U}^{(2)}_c$ in the limit $n a_0^d \to 0$ for $m_1 = m_2$. In both cases, $\mathcal{U}^{(d)}_c$ approaches the mean-field value $u_{\rm mf} = \sqrt{u_{11}u_{22}}$ at high density. We notice that the action S [eq. (27)] possesses an SU(2) symmetry when $m_1 = m_2$, $\mu_1 = \mu_2$, and $u_{11} = u_{22} = u_{12}$. In this SU(2) symmetrical limit, $u_c = u_{\rm mf}$, and the phase boundary may coincide with the mean-field prediction.

4. Conclusions and Discussions

We employ the RG to study the dilute BF and BB mixtures with repulsive interactions in low dimensions. For the 1D BF mixture, we show that the uniform mixture is stable against de-mixing for any finite strength of repulsions. This conclusion is consistent with the prediction obtained from the Bethe ansatz when the system can be solved exactly. The bosonization approach gave a stability condition of the mean-field type at large boson densities (the weak-coupling regime),[6] while at low boson densities (the strong-coupling regime) the stability condition becomes $0 < u_{12} < \pi^2\sqrt{n_b n_f/(m_b m_f)}$. This result is inconsistent with the exact solution, while in the solvable limit ours is consistent with the Bethe-ansatz result. We believe that this inconsistency arises from the fact that the relations between the Luttinger liquid parameters and the short-distance parameters like g_{bb} and g_{bf} are not reliable in the dilute limit where the system goes into the strong-coupling regime.

For the 2D BF mixture with repulsions in the extremely dilute limit, the uniform mixture is also stable irrespective of the interaction strength. The mean-field result is expected to be valid in the moderate and high density regime. We also calculate the critical temperature T_c for the SF phase. By comparing with T_c for the single-component dilute Bose gas, we find that the inclusion of fermions increases T_c. That is, the presence of fermions enhances the long range SF ordering.

For the 1D BB mixture, the low-energy physics is controlled by a fixed point which does not support the uniform mixture. This observation immediately leads to two consequences: (i) First of all, the equilibrium properties of the uniform mixture will not exhibit universal behaviors even in the strong-coupling regime because we cannot take the continuum limit. (ii) Next, the region of the stable uniform mixture with $m_1 \neq m_2$ can be very narrow in the dilute limit. For the 2D BB mixture in the extremely dilute limit, the region of the stable uniform mixture with $m_1 \neq m_2$ is also narrow because the interactions are strongly renormalized in this limit. The mean-field result can be applied only in the moderate and high density regime.

A recent paper also studied the stability conditions for the low-dimensional dilute BB mixture in terms of the RG approach.[16] The main distinction between our results and the ones in ref. 16 lies at the 1D case with $m_1 \neq m_2$, where our predicted value of $\mathcal{U}^{(1)}_c$ can be much smaller than u_c in the dilute limit (our u_c is basically $u^{(1d)}$ in ref. 16), while in one dimension with $m_1 = m_2$ and in two dimensions, both are qualitatively consistent with each other. The discrepancy may be originated from the different criteria of the stability: In ref. 16, the criterion for the stability is the positive definiteness of the potential term in the action at any scale $l < l^*$. In our opinion, the use of the positive definiteness of the potential term in the action amounts to some kind of mean-field treatment, and such a procedure is meaningful only when the quantum fluctuations have been properly taken into account. Hence, this criterion can be used only at the scale $l = l^*$ where the correlation length becomes of $O(1)$ in the unit of the short-distance cutoff at $l = l^*$. This is the criterion which we employed in this work. The issue about whether or not the critical strength for the 1D uniform BB mixture can be very small in the dilute limit should be answered by further numerical or experimental studies.

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Appendix: Mean-Field Theory of the Uniform Bose–Fermi Mixture in d Dimensions

By taking the ansatz $\rho = |\phi|^2$, The mean-field free energy density f for the action S [eq. (1)] is of the form

$$f = -\mu_b \rho + \frac{g_{bb}}{2}\rho^2 - \frac{2K_d(2m_f)^{d/2}}{d(d+2)}\Theta(\mu_f - g_{bf}\rho)(\mu_f - g_{bf}\rho)^{d/2+1},$$

where $\Theta(x) = 0, 1$ for $x < 0$ and $x > 0$, respectively. The value of ρ is determined by the minimum of f, which leads to the mean-field equation

$$\frac{K_d g_{\mathrm{bf}}}{d}\,\Theta(\mu_{\mathrm{f}} - g_{\mathrm{bf}}\rho)[2m_{\mathrm{f}}(\mu_{\mathrm{f}} - g_{\mathrm{bf}}\rho)]^{d/2} = \mu_{\mathrm{b}} - g_{\mathrm{bb}}\rho.$$

The uniform mixture corresponds to the solution of the mean-field equation with $0 < \rho < \mu_{\mathrm{f}}/g_{\mathrm{bf}}$. Hence, for the uniform mixture, the free energy density is of the form

$$f_{\mathrm{M}} = -\mu_{\mathrm{b}}\rho + \frac{g_{\mathrm{bb}}}{2}\rho^2 - \frac{2K_d(2m_{\mathrm{f}})^{d/2}}{d(d+2)}(\mu_{\mathrm{f}} - g_{\mathrm{bf}}\rho)^{d/2+1}, \tag{A·1}$$

where ρ_{b} is the solution of the equation

$$\frac{K_d g_{\mathrm{bf}}}{d}[2m_{\mathrm{f}}(\mu_{\mathrm{f}} - g_{\mathrm{bf}}\rho)]^{d/2} = \mu_{\mathrm{b}} - g_{\mathrm{bb}}\rho, \tag{A·2}$$

satisfying the constraint $0 < \rho < \mu_{\mathrm{f}}/g_{\mathrm{bf}}$. The density of fermions ρ_{f} can be obtained from f_{M} through the thermodynamical relation $\rho_{\mathrm{f}} = -\partial f_{\mathrm{M}}/\partial\mu_{\mathrm{f}}$, yielding

$$\rho_{\mathrm{f}} = \frac{K_d}{d}[2m_{\mathrm{f}}(\mu_{\mathrm{f}} - g_{\mathrm{bf}}\rho)]^{d/2}. \tag{A·3}$$

Using the thermodynamical relation $\rho_{\mathrm{b}} = -\partial f_{\mathrm{M}}/\partial\mu_{\mathrm{b}}$ where ρ_{b} denotes the boson density, one may find that $\rho = \rho_{\mathrm{b}}$. In terms of this relation and eqs. (A·2), (A·3) to eliminate μ_α, one may express f_{M} as a function of ρ_α. Using Euler's relation $f_{\mathrm{M}} = -P$ and the thermodynamical relation $P = -\partial E/\partial V$, one may obtain the total energy density

$$\frac{E_{\mathrm{M}}}{V} = \frac{g_{\mathrm{bb}}}{2}\rho_{\mathrm{b}}^2 + g_{\mathrm{bf}}\rho_{\mathrm{b}}\rho_{\mathrm{f}} + \frac{d}{d+2}g_{\mathrm{ff}}\rho_{\mathrm{f}}^{1+2/d}, \tag{A·4}$$

where

$$g_{\mathrm{ff}} = \frac{1}{2m_{\mathrm{f}}}\left(\frac{d}{K_d}\right)^{2/d}.$$

By considering small density fluctuations, we obtain from eq. (A·4) the linear stability condition

$$g_{\mathrm{bf}}^2 < \frac{2g_{\mathrm{ff}}}{d}g_{\mathrm{bb}}n_{\mathrm{f}}^{2/d-1}. \tag{A·5}$$

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20) Strictly speaking, we should define l_* by $r_{\mathrm{b}}(l_*) + r_{\mathrm{f}}(l_*) = \alpha$ where $\alpha = O(1)$. However, α always appears in e^{l_*} in the guise of a_0/α. This means that the exact value of α depends on the choice of the short-distance cutoff a_0, which does not affect our results, such as the dependence of the physical quantities on n_{b} and n_{f}.
21) The condition $e^{\epsilon l_*} \gg 1$ insures that we are adressing the physics near the interacting zero-density fixed point, i.e., the low-energy physics in the strong-coupling regime. In particular, we obtain eqs. (16) or (17) by assuming $e^{-\epsilon l_*} \ll 1$. Hence, a naive substitution of $\epsilon = 0$ into them will give rise to $e^{\epsilon l_*} \to 1$ and contradict this condition. Therefore, these results in $2 - \epsilon$ dimensions can not be used in 2D, which has a completely different fixed-point structure.
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